The Annihilating-Ideal Graph of Commutative Rings II*

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Abstract

In this paper we continue our study of annihilating-ideal graph of commutative rings, that was introduced in Part I (see [4]). Let R be a commutative ring with $\mathbb{A}(R)$ its set of ideals with nonzero annihilator and $Z(R)$ its set of zero divisors. The annihilating-ideal graph of R is defined as the (undirected) graph $\mathbb{AG}(R)$ that its vertices are $\mathbb{A}(R)^* = \mathbb{A}(R) \setminus \{(0)\}$ in which for every distinct vertices I and J, I —— J is an edge if and only if $IJ = (0)$. First, we study the diameter of $\mathbb{AG}(R)$. A complete characterization for the possible diameter is given exclusively in terms of the ideals of R when either R is a Noetherian ring or $Z(R)$ is not an ideal of R. Next, we study coloring of annihilating-ideal graphs. Among other results, we characterize when either $\chi(\mathbb{AG}(R)) \leq 2$ or R is reduced and $\chi(\mathbb{AG}(R)) \leq \infty$. Also it is shown that for each reduced ring R, $\chi(\mathbb{AG}(R)) = cl(\mathbb{AG}(R))$. Moreover, if $\chi(\mathbb{AG}(R))$ is finite, then R has a finite number of minimal primes, and if n is this number, then $\chi(\mathbb{AG}(R)) = cl(\mathbb{AG}(R)) = n$. Finally, we show that for a Noetherian ring R, $cl(\mathbb{AG}(R))$ is finite if and only if for every ideal I of R with $I^2 = (0)$, I has finite number of R-submodules.

Key Words: Commutative rings; Annihilating-ideal; Zero-divisor; Graph; Coloring of graphs

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0. Introduction

The present paper is a sequel to [4] and so the notations introduced in Introduction of [4] will remain in force. Thus throughout the paper, R denotes a commutative ring with identity, $Z(R)$ denotes the the set of all zero divisors of R and $\mathbb{I}(R)$ denotes the set of all proper ideals of R. If X is either an element or a subset of R, then the annihilator of X is $Ann(X) = \{r \in R | rX = 0\}$. The zero divisor graph of R, denoted by $\Gamma(R)$, is a graph with the vertex set $Z(R)^* := Z(R)$ in which for every two vertices x and y, x ——— y is an edge if and only if $x \neq y$ and $xy = 0$. As [4], we say that the ideal I of R is an *annihilating-ideal* if $Ann(I) \neq (0)$ (i.e., there exists a nonzero ideal J of R such that $IJ = (0)$). Let $\mathbb{A}(R)$ be the set of all annihilating-ideals of R. Then the *annihilating-ideal graph* of R, denoted by $\mathbb{AG}(R)$, is a undirected simple graph with the vertex set $\mathbb{A}(R)^* := \mathbb{A}(R) \setminus \{(0)\}$ in which every two distinct vertices I and J are adjacent if and only if $IJ = (0)$ (see Part I [4] for more details).

Recall that a graph G is connected if there is a path between every two distinct vertices. For distinct vertices x and y of G, let $d(x, y)$ be the length of the shortest path from x to y and if there is no such path we define $d(x, y) = \infty$. The diameter of G is $diam(G) = sup\{d(x, y) : x \ and \ y \ are \ distinct \ vertices \ of \ G\}$. The girth of G, denoted by $g(G)$, is defined as the length of the shortest cycle in G and $g(G) = \infty$ if G contains no cycles. A graph in which each pair of distinct vertices is joined by an edge is called a complete graph. Also, if a graph G contains one vertex to which all other vertices are joined and G has no other edges, is called a star graph. In [2, Theorem 2.1], it is shown that for every ring R, $\mathbb{AG}(R)$ is a connected graph and $diam(\mathbb{AG})(R) \leq 3$, and if $\mathbb{AG}(R)$ contains a cycle, then $g(\mathbb{AG}(R)) \leq 4$ (see [7]). In Section 1 of this paper, we study the diameter of the annihilating-ideal graphs. By using the papers [4] and [6], we determine the relationship between the diameter of $\mathbb{AG}(R)$ and $\Gamma(R)$. In particular, a complete characterization for the possible diameter is given exclusively in terms of the ideals of R when either R has finitely many minimal primes or $Z(R)$ is not an ideal.

A clique of a graph is its maximal complete subgraph and the number of vertices in the largest clique of graph G, denoted by $cl(G)$, is called the clique number of G. Let $\chi(G)$ denote the chromatic number of the graph G, that is, the minimal number of colors needed to color the vertices of G so that no two adjacent vertices have the same color. Obviously $\chi(G) \geq cl(G)$. Beck in [3] conjectured that $\chi(\Gamma(R)) = cl(\Gamma(R))$. But, D.D. Anderson and M. Naseer

gave a counterexample to this conjecture in [1]. In fact, the counterexample is the ring $R = \mathbb{Z}_4[X,Y,Z]/(X^2-2, Y^2-2, Z^2, 2X, 2Y, 2Z, XY, XZ, YZ-2)$ for which $\chi(\Gamma(R)) = 5$ but $cl(\Gamma(R)) = 4$ (note that in Beck's coloring all elements of the ring were vertices of the graph but the vertices of $\Gamma(R)$ are nonzero zero divisors of R). In Section 2, we look at the coloring of the annihilating-ideal graph of rings. First, we show that for the counterexample above we have $\chi(\mathbb{AG}(R)) = cl(\mathbb{AG}(R)) = 4$. On the other hand, however we have not found any example where $\chi(\mathbb{AG}(R)) > cl(\mathbb{AG}(R))$. The lack of such counterexamples together with the fact that we have been able to establish the equality $\chi(\mathbb{AG}(R)) = cl(\mathbb{AG}(R))$ for Anderson-Naseer's counterexample and also for reduced rings motivates the following conjecture.

Conjecture 0.1. *For every commutative ring R, $\chi(\mathbb{AG}(R)) = cl(\mathbb{AG}(R))$.*

In Section 2, among other results, we characterize rings R for which $\chi(\mathbb{AG}(R)) \leq 2$. It is shown that for a reduced ring R the following conditions are equivalent: (1) $\chi(\mathbb{AG}(R)) < \infty$, (2) $cl(\mathbb{AG}(R)) < \infty$, (3) $\mathbb{AG}(R)$ does not have an infinite clique and (4) R has finite number of minimal primes. Moreover, if R a non-domain reduced ring, then $\chi(\mathbb{AG}(R)$ is the number of minimal primes of R. Also, it is shown that for a Noetherian ring R, $cl(\mathbb{AG}(R)$ is finite if and only if every ideal I of R with $I^2 = (0)$ has finite number of R-submodules. Finally we conjecture that "if $\mathbb{AG}(R)$ does not have an infinite clique, then $\chi(\mathbb{AG}(R))$ is finite".

1. The diameter of an annihilating-ideal graph

By Anderson and Livingston [2, Theorem 2.3], for every ring R, the zero divisor graph $\Gamma(R)$ is a connected graph and $diam(\Gamma(R)) \leq 3$. Moreover, if $\Gamma(R)$ contains a cycle, then g($\Gamma(R)) \leq 4$ (see [7]). Moreover, Lucas in [6] characterized the diameter of $\Gamma(R)$ in terms of the ideals of R. As we have seen in [4, Theorem 2.1], for a ring R, $\mathbb{AG}(R)$ is also a connected graph with $diam(\mathbb{AG}(R) \leq 3$. In general, Lucas's results are not true for annihilating-ideal graphs, but the following proposition more or less summarizes the overall situation for the relationship between the diameter of $\mathbb{AG}(R)$ and $\Gamma(R)$.

Proposition 1.1. *Let R be a ring.*

(a) *If $diam(\Gamma(R)) = 0$, then $diam(\mathbb{AG}(R)) = 0$.*
(b) *If $diam(\Gamma(R)) = 1$, then $diam(\mathbb{AG}(R)) = 0$ or 1.*
(c) *If $diam(\Gamma(R)) = 2$, then $diam(\mathbb{AG}(R)) = 1, 2$ or 3.*

(d) *If* $diam(\Gamma(R)) = 3$*, then* $diam(\mathbb{AG}(R)) = 3$*.*
(e) *If* $diam(\mathbb{AG}(R)) = 0$*, then* $diam(\Gamma(R)) = 0$ *or* 1*.*
(f) *If* $diam(\mathbb{AG}(R)) = 1$*, then* $diam(\Gamma(R)) = 1$ *or* 2*.*
(g) *If* $diam(\mathbb{AG}(R)) = 2$*, then* $diam(\Gamma(R)) = 2$*.*
(h) *If* $diam(\mathbb{AG}(R)) = 3$*, then* $diam(\Gamma(R)) = 2$ *or* 3*.*

Proof. (a). Let $diam(\Gamma(R)) = 0$ i.e., $\Gamma(R)$ has one vertex. Thus by [6, Theorem 2.6.], R is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2[y]/(y^2)$. Clearly, in any case $\mathbb{AG}(R)$ has also one vertex and so $diam(\mathbb{AG}(R)) = 0$.
(b). Clearly, if $diam(\Gamma(R)) = 1$ then $\Gamma(R)$ is a complete graph with more than one vertex. Thus by [6, Theorem 2.6], either (i) R is reduced and isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, or (ii) R is non-reduced, $Z(R)^2 = (0)$ and R is not isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2[y]/(y^2)$. If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $diam(\mathbb{AG}(R)) = 1$. In the later case, $Z(R)^2 = (0)$ implies that $\mathbb{AG}(R)$ is also a complete graph (see [4, Theorem 2.7]. Now, if $\mathbb{AG}(R)$ has one vertex (as $\mathbb{Z}_{p^2}$ where p is an odd prime number), then $diam(\mathbb{AG}(R)) = 0$, otherwise, $diam(\mathbb{AG}(R)) = 1$.
(c). By [6, Theorem 2.6.], $diam(\Gamma(R)) = 2$ if and only if either R is reduced with exactly two minimal primes and at least three nonzero zero divisors, or $Z(R)$ is an ideal whose square is not (0) and each pair of distinct zero divisors has a nonzero annihilator. If $diam(\mathbb{AG}(R)) = 0$, then $\mathbb{AG}(R)$ has only one vertex, say I, and $I^2 = (0)$. It follows that $Z(R) = I$ and so by [2, Theorem 2.8], $\Gamma(R)$ is a complete graph. Thus $diam(\Gamma(R)) = 0$ or 1, contradicting with our hypothesis. Thus $diam(\mathbb{AG}(R)) > 0$, i.e., $diam(\mathbb{AG}(R)) = 1, 2$ or 3 (since by [4, Theorem 2.1], $diam(\mathbb{AG}(R)) \leq 3$). We claim that all these three cases may happen. For see this, let $R_2 = F_1 \times F_2$ where F_1 and F_2 are fields and $F_1 \ncong \mathbb{Z}_2$ and $R_1 = \mathbb{Z} \times \mathbb{Z}_2$. Then $diam(\Gamma(R_1)) = diam(\Gamma(R_2)) = 2$ but $diam(\mathbb{AG}(R_1)) = 1$ and $diam(\mathbb{AG}(R_2)) = 2$. For an example of a ring R for which $diam(\Gamma(R)) = 2$ but $diam(\mathbb{AG}(R_2)) = 3$ see the Example 1.7 of this paper.
(d). Let $diam(\Gamma(R)) = 3$. Then there exist $a, b \in Z(R)$ such that $d(a, b) = 3$. Clearly, Ra and Rb are vertices of $\mathbb{AG}(R)$ and $RaRb \neq (0)$. We claim that $Ra \neq Rb$, for if not, then $Ra = Rb$ implies that a and b have the same annihilator i.e., $d(a, b) = 2$, a contradiction. Now by [4, Theorem 2.1], $d(Ra, Rb) \leq 3$ in $\mathbb{AG}(R)$. If $d(Ra, Rb) \neq 3$, then $d(Ra, Rb) = 2$, i.e., there exists $0 \neq c \in R$ such that $RcRa = RcRb = (0)$. It follows $d(a, b) = 2$ in $\Gamma(R)$, a contradiction. Thus $d(Ra, Rb) = 3$ and so $diam(\mathbb{AG}(R)) = 3$.
(e). Clearly, $diam(\mathbb{AG}(R)) = 0$ if and only if the ring R has only one nonzero proper ideal, if and only if $Z(R)$ is the only nonzero proper ideal of R. In this case $Z(R)^2 = (0)$ and so $\Gamma(R)$ is a complete graph. If $\Gamma(R)$ has one vertex, then by [6, Theorem 2.6 (1)], R is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2[y]/(y^2)$

and so $diam(\Gamma(R)) = 0$, otherwise, when $\Gamma(R)$ has more than one vertices (as $R = \mathbb{Z}_{p^2}$ where p is an odd prime number), then $diam(\Gamma(R)) = 1$.
(f). Clearly, $diam(\mathbb{AG}(R)) = 1$ if and only if $\mathbb{AG}(R)$ is a complete graph with more than one vertex, i.e., either $R = F_1 \oplus F_2$, where F_1, F_2 are fields, or $Z(R)$ is an ideal of R that is not minimal, $Z(R)^3 = (0)$ and for each ideal $I \subsetneqq Z(R)$, $IZ(R) = (0)$ (see [4, Theorem 2.7]). If $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $Z(R)^2 = (0)$, then $\Gamma(R)$ is a complete graph with more than one vertex i.e., $diam(\Gamma(R)) = 1$. On the other hand, if $R \ncong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $Z(R)^2 \neq (0)$, then $\Gamma(R)$ is not a complete graph and so $diam(\Gamma(R)) \geq 2$. If $diam(\Gamma(R)) = 3$, then by (d) above $diam(\mathbb{AG}(R)) = 3$, a contradiction. Thus we have $diam(\Gamma(R)) = 2$.
(g). If $diam(\mathbb{AG}(R)) = 2$, then by (a), (b) and (d) above, $diam(\Gamma(R)) \neq 0,\ 1$ and 3. Thus $diam(\Gamma(R)) = 2$.
(h). If $diam(\mathbb{AG}(R)) = 3$, then by (a) and (b) above, $diam(\Gamma(R)) \neq 0$ and 1. Thus $diam(\Gamma(R)) = 2$ or 3. By (d) above if $diam(\Gamma(R)) = 3$, then $diam(\mathbb{AG}(R))$ is also 3. For the case $diam(\mathbb{AG}(R)) = 3$ but $diam(\Gamma(R)) = 2$, see Example 1.7. □

In next two theorems, we characterize when $Z(R)$ is not an ideal of R and either $diam(\mathbb{AG}(R)) = 2$ or $diam(\mathbb{AG}(R)) = 3$, respectively.

Theorem 1.2. *Let R be a ring such that $Z(R)$ is not an ideal of R. Then $diam(\mathbb{AG}(R)) = 2$ if and only if R is reduced with exactly two minimal primes and at least three nonzero annihilating ideals.*

Proof. ($\Rightarrow$). Let $diam(\mathbb{AG}(R)) = 2$. Then R has at least three nonzero annihilating ideals and by Proposition 1.1 (g), $diam(\Gamma(R)) = 2$. Now by [6, Theorem 2.6, (3)], R is a reduced ring with exactly two minimal primes.
($\Leftarrow$). Suppose R is a reduced ring with exactly two minimal primes P_1 and P_2. Let $D_1 := R/P_1$, $D_2 := R/P_2$. Since R is reduced, $P_1 \cap P_2 = (0)$. Thus the neutral map $\varphi : R \longrightarrow D_1 \times D_2$ is a monomorphism ring homomorphism. It is easy to check that if I and J are vertices in $\mathbb{AG}(R)$ such that $IJ = (0)$, then $\bar{I} := \varphi(I)$ and $\bar{J} := \varphi(J)$ are also vertices of $\mathbb{AG}(D_1 \times D_2)$ and $\bar{I}\bar{J} = (0)$. Since D_1 and D_2 are domains, each vertex of $\mathbb{AG}(D_1 \times D_2)$ is the form $(A, (0)$ or $((0), B)$ for some nonzero ideals A of D_1 or B of D_2. First, we show that $diam(\mathbb{AG}(R)) \leq 2$. For see this, suppose I and J are two distinct vertices in $\mathbb{AG}(R)$. We need only consider 2 cases:
Case 1: $\varphi(I) = (\bar{I}, (\bar{0}))$ and $\varphi(J) = (\bar{J}, (\bar{0}))$. It follows that $I,\ J \subseteq P_2$. Since $P_1P_2 = (0)$, $P_1I = P_1J = (0)$ and so $d(I, J) \leq 2$.
Case 2: $\varphi(I) = (\bar{I}, (\bar{0}))$ and $\varphi(J) = ((\bar{0}), \bar{J})$. It follows that $I \subseteq P_2$ and

$J \subseteq P_1$. Since $P_1P_2 = (0)$, $IJ = (0)$ and so $d(I, J) = 1$.
Thus $diam(\mathbb{AG}(R)) \leq 2$. Since P_1 and P_2 are two distinct vertices of $\mathbb{AG}(R)$, $diam(\mathbb{AG}(R)) \geq 1$. If $diam(\mathbb{AG}(R)) = 1$, then for each nonzero ideal I of R such that $P_1 \neq I \neq P_2$, we must have $IP_1 = IP_2 = (0)$. It follows that $I \subseteq P_1 \cap P_2 = (0)$, a contradiction. Thus $diam(\mathbb{AG}(R)) = 2$. □

Theorem 1.3. *Let R be a ring such that $Z(R)$ is not an ideal. Then the following statements are equivalents.*

(1) $diam(\mathbb{AG}(R)) = 3$.
(2) $diam(\Gamma(R)) = 3$.
(3) *Either R is non-reduced or R is a reduced ring with more than two minimal primes.*

Proof. (1) $\Rightarrow$ (2). Let $diam(\mathbb{AG}(R)) = 3$. If $diam(\Gamma(R)) \neq 3$, then by Proposition 1.1 (h), $diam(\Gamma(R)) = 2$. Thus by [6, Theorem 2.6, (3)], R is reduced with exactly two minimal primes. Since $diam(\mathbb{AG}(R)) = 3$, there exist at least three nonzero annihilating ideals in R. Thus by Theorem 1.2, $diam(\mathbb{AG}(R)) = 2$, a contradiction. Therefore, $diam(\Gamma(R)) = 3$.
(2) $\Rightarrow$ (1) is by Proposition 1.1, (d).
(2) $\Leftrightarrow$ (3) is by [6, Theorem 2.6.]. □

Now, we are in position to give a complete characterization for the possible diameter of $\mathbb{AG}(R)$ when $Z(R)$ is not an ideal of R.

Theorem 1.4. *Let R be a ring such that $Z(R)$ is not an ideal of R. Then $1 \leq diam(\mathbb{AG}(R) \leq 3$ and*

(1) $diam(\mathbb{AG}(R)) = 1$ *if and only if* $R \cong F_1 \times F_2$ *where* F_1 *and* F_2 *are fields;*
(2) $diam(\mathbb{AG}(R)) = 2$ *if and only if R is reduced with exactly two minimal primes and at least three nonzero annihilating ideals;*
(3) $diam(\mathbb{AG}(R)) = 3$ *if and only if either R is a non-reduced ring or R is a reduced ring with more than two minimal primes.*

Proof. Suppose that $Z(R)$ is not an ideal of R. Then by [4, Theorem 2.1], $diam(\mathbb{AG}(R)) \leq 3$. If $diam(\mathbb{AG}(R) = 0$, then $\mathbb{AG}(R)$ has only one vertex, i.e., $Z(R)$ is the only nonzero proper ideal of R, a contradiction. Thus we conclude that $diam(\mathbb{AG}(R) = 1,\ 2$ or 3.
For (1), we note that $\mathbb{AG}(R)$ is a complete graph (i.e., $diam(\mathbb{AG}(R)) = 0$ or 1) if and only if either $R = F_1 \oplus F_2$, where F_1, F_2 are fields, or $Z(R)$ is an

ideal of R, $Z(R)^3 = (0)$ and for each ideal $I \subsetneqq Z(R)$, $IZ(R) = (0)$ (see [4, Theorem 2.7]). Thus $diam(\mathbb{AG}(R)) = 1$ if and only if $R \cong F_1 \times F_2$ where F_1 and F_2 are fields.
Finally statements (2) and (3) are from Theorems 1.2 and 1.3 above. □

In the next theorem, we provide a sufficient condition for $\mathbb{AG}(R)$ to have diameter 3 when R is a non-reduced ring. First we need the following lemma (see also [6, Lemma 2.3]).

Lemma 1.5. *Let R be a ring and let I be an annihilating-ideal of R. If N is a nilpotent ideal of R, then $N + I$ is an annihilating-ideal of R.*

Proof. Let N be a nonzero nilpotent ideal and assume $cI = (0)$ where $c \neq 0$. Since N is nilpotent, there is a positive integer m such that $cN^m = 0$ with $cN^{m-1} \neq 0$. Clearly, $cN^{m-1} \subseteq Ann(N + I)$. □

Theorem 1.6. *Let R be a non-reduced ring. If there is a pair of annihilating-ideals I, J of R such that $I+J$ is not an annihilating-ideal, then $diam(\mathbb{AG}(R)) = 3$.*

Proof. Let $I, J \in \mathbb{A}^*(R)$ be such that $Ann(I + J) = (0)$. Then $d(I, J) \neq 2$. By Lemma 1.5, neither I nor J can be nilpotent. If $IJ \neq (0)$, then $d(I, J) = 3$. Thus we may assume $IJ = (0)$. Since $IJ = (0)$, $(I + J)^2 = I^2 + J^2$ has no nonzero annihilator. Since R is not reduced, there exists a nonzero nilpotent $q \in R$. Thus without loss of generality we may assume that $qJ^2 \neq (0)$. Since I is an annihilating-ideal and qJ is nilpotent, $I + qJ$ is an annihilating-ideal by Lemma 1.5. On the other hand $I + qJ \neq J$, for if not, then $I \subseteq J$ and so $Ann(I + J) = Ann(J) \neq (0)$, a contradiction. Now consider the pair $I + qJ$ and J. Since $I + J = I + qJ + J$ and $Ann(I + J) = (0)$, $d(I + qJ, J) \neq 2$. But $(I + qJ)J = qJ^2 \neq (0)$. Thus $d(I + qJ, J) = 3$ and $diam(\mathbb{AG}(R)) = 3$. □

Example 1.7. Let R be the ring in [6, Example 5.5.]. Then R is a non-reduced ring and there is a pair of annihilating-ideals I, J of R such that $I + J$ is not an annihilating-ideal (see [6, Example 5.5, (6)]. Thus by Theorem 1.6, $diam(\mathbb{AG}(R)) = 3$, but since R is a McCoy ring (see [6, Example 5.5, (4)] $diam(\Gamma(R)) = 2$. (Note: a ring R is said to be a McCoy ring if each finitely generated ideal contained in $Z(R)$ has a nonzero annihilator).

Let R be a ring with finitely many minimal primes $P_1, P_2, \cdots P_n$. Then it is easy to see that an ideal I of R is an annihilating-ideal if and only if I is contained in at least one minimal prime.

Lemma 1.8. *Let R be a ring with finitely many minimal primes. If R has more two minimal primes and there are nonzero annihilating-ideals I, J such that $I+J$ is not an annihilating-ideal, then $diam(\mathbb{AG}(R)) = 3$.*

Proof. By Theorem 1.6, we can assume that R is a reduced ring. If $IJ \neq (0)$, then then $d(I, J) = 3$ and therefore $diam(\mathbb{AG}(R)) = 3$. Thus we may assume R has more than two minimal primes and that there are nonzero annihilating-ideals I and J, such that $IJ \neq (0)$ and $Ann(I + J) = (0)$. Each minimal prime of R contains at least one of I and J, but with $Ann(I+J) = (0)$, and not both of them. Thus without loss of generality we may assume there are minimal primes P_1, P_2 and P_3 such that $I \subseteq P_1 \cap P_2$, $I \nsubseteq P_3$, $J \subseteq P_3$, $J \nsubseteq P_1$ and $J \nsubseteq P_2$. Let $q \in P_1 \cap P_3 \backslash P_2$ and consider the pair $I + qJ$ and J. Since R is reduced, $IJ = (0)$ and neither I nor q is contained in P_2, $(0) \neq qJ^2 = J(I + qJ)$. Since $I + qJ \subseteq P_1$, that $I + qJ$ is a nonzero annihilating-ideal of R. On the other hand, $I + J = I + qJ + J$ is an ideal with no nonzero annihilator. Thus $d(I+qJ, J) = 3$ and $diam(\mathbb{AG}(R)) = 3$. $\square$

Now, we are in position to give a complete characterization for the possible diameter of $\mathbb{AG}(R)$ when R has finitely many minimal primes (as Noetherian rings).

Theorem 1.9. *Let R be a ring with finitely many minimal primes. Then*

(1) *$diam(\mathbb{AG}(R)) = 0$ if and only if $Z(R)$ is the only nonzero proper ideal of R.*

(2) *$diam(\mathbb{AG}(R)) = 1$ if and only if $Z(R)$ is an ideal of R, $Z(R)^3 = (0)$ and for each ideal $I \subsetneqq Z(R)$, $IZ(R) = (0)$.*

(3) *$diam(\mathbb{AG}(R)) = 2$ if and only if either* (i) *R is reduced with exactly two minimal primes and at least three nonzero annihilating-ideals, or* (ii) *$Z(R)$ is an ideal whose square is not* (0) *and for each pair of annihilating-ideals I and J, $I + J$ is an annihilating-ideal.*

(4) *$diam(\mathbb{AG}(R)) = 3$ if and only if there are annihilating-ideals $I \neq J$, such that $I + J$ is not an annihilating-ideal and either* (i) *R is reduced with more than two minimal primes, or* (ii) *R is non-reduced.*

Proof. (1) is evident.
The statement (2) is from [4, Theorem 2.7].
(3) ($\Rightarrow$). Suppose that $diam(\mathbb{AG}(R)) = 2$. Then by Proposition 1.1 $diam(\Gamma(R)) =$

2. Thus by [6, Theorem 2.6 (3)], either (i) R is reduced with exactly two minimal primes and at least three nonzero zero divisors, or (ii) $Z(R)$ is an ideal whose square is not (0) and each pair of distinct zero divisors has a nonzero annihilator. The Case (i) implies that $Z(R)$ is not an ideal of R and so by Theorem 1.2, R is a reduced ring with exactly two minimal primes and at least three nonzero annihilating-ideals. Assume that the Case (ii) holds. If R is not reduced, then by Theorem 1.6, for each pair of annihilating-ideals I and J, $I + J$ is an annihilating-ideal. Thus we can assume that R is reduced and R has more than two minimal primes. Hence by Lemma 1.8, for each pair of annihilating-ideals I and J, $I + J$ is an annihilating-ideal.
(3) ($\Leftarrow$) is evident.
Finally statement (4) is from Theorems 1.3, 1.6, Lemma 1.8 and statement (3) above. $\square$

2. Coloring of the annihilating ideal graphs

The goal of this section is to study of coloring of the annihilating ideal graphs of rings, in particular the interplay between $\chi(\mathbb{AG}(R))$ and $cl(\mathbb{AG}(R))$.

Beck [3] showed that if R is a reduced ring or a principal ideal ring, then $\chi(\Gamma(R)) = cl(\Gamma(R))$ He also shoed that for $n = 1, 2$ or 3, $\chi(\Gamma(R)) = n$ if and only if $cl(\Gamma(R)) = n$. Based on these positive results, Beck conjectured that $\chi(\Gamma(R)) = cl(\Gamma(R))$ for each ring R when $\chi(\Gamma(R)) < \infty$. In [1], D.D. Anderson and M. Naseer gave a counterexample to this conjecture. We start this section with the following interesting result abut Anderson-Naseer's counterexample.

Proposition 2.1. *Let*

$$R = \mathbb{Z}_4[X, Y, Z]/(X^2 - 2, Y^2 - 2, Z^2, 2X, 2Y, 2Z, XY, XZ, YZ - 2).$$

Then $\chi(\mathbb{AG}(R)) = cl(\mathbb{AG}(R)) = 4.$

Proof. Clearly, R is a finite local ring with 32 elements and $J(R) = Z(R) = \{0, 2, x + 2, y, y + 2, x + y, x + y + 2, z, z + 2, x + z, x + z + 2, y + z, y + z + 2, x + y + z, x + y + z + 2\}$. One can easily see that the following 15 ideals are the only nonzero proper ideals of R.
$\{(2), (x), (y), (z), (x + y), (x + z), (y + z), (x + y + z), (x, y), (x, z), (y, z), (x, y + z), (y, x + z), (z, x + y), (x, y, z)\}$.
Thus the graph $\mathbb{AG}(R)$ has 15 vertices and so is the graph in Figure 1 below.

Now by using Figure 1, it easy to see that $\{(2),(x),(y),(y+z)\}$ is a maximal clique and each other clique of $\mathbb{AG}(R)$ has at most 3 elements. Thus $cl(\mathbb{AG}(R)) = 4$. Also, the Figure 1, shows that the minimal number of colors needed to color $\mathbb{AG}(R)$ is 4 colors, which we label as 1, 2, 3 and 4. Thus $\chi(\mathbb{AG}(R)) = cl(\mathbb{AG}(R)) = 4$. $\square$

Next, we characterize rings R for which $\chi(R) = 1$ or 2.

Proposition 2.2. *Let R be a ring. Then $\chi(\mathbb{AG}(R)) = 1$ if and only if R has only one nonzero proper ideal.*

Proof. Since $\mathbb{AG}(R)$ is a connected graph and $\chi(\mathbb{AG}(R)) = 1$, it can not have more than one vertex, and so by [4, Theorem 1.4], R has only one nonzero proper ideal. The converse is clear. $\square$

Theorem 2.3. *For a ring R the following statements are equivalent:*

(1) $\chi(\mathbb{AG}(R)) = 2$.
(2) $\mathbb{AG}(R)$ *is a bipartite graph.*
(3) $\mathbb{AG}(R)$ *is a complete bipartite graph.*
(4) *Either R is a reduced ring with exactly two minimal primes or $\mathbb{AG}(R)$ is a star graph.*

Proof. (1) $\Leftrightarrow$ (2) and (3) $\Rightarrow$ (2) are clear.
(2) $\Rightarrow$ (4). Suppose that $\mathbb{AG}(R)$ is bipartite with two parts V_1 and V_2.
Case 1: The ring R is a reduced ring. We put $P_1 = \bigcup_{I \in V_1} I, P_2 = \bigcup_{J \in V_2} J$. We claim that P_1 and P_2 are prime ideals of R. Let $x \in P_1$ and $r \in R$. Then there exists $I \in V_1$ such that $Rx \subseteq I$. Since $I \in V_1$ and $\mathbb{AG}(R)$ is bipartite, there exists $J \in V_2$ such that $IJ = (0)$. Thus $RrxJ = (0)$ and so $Rrx \in V_1$, and hence $rx \in P_1$. Now let $x,\ y \in P_1$. Then there exist $I_1,\ I_2 \in V_1$ and $J_1,\ J_2 \in V_2$ such that $x \in I_1$, $y \in I_2$ and $I_1J_1 = I_2J_2 = (0)$. Thus $RxJ_1 = RyJ_2 = (0)$. Since $\mathbb{AG}(R)$ is bipartite and R is reduced, $J_1J_2 \neq (0)$. On the other hand, $RxJ_1J_2 = RyJ_1J_2 = (0)$ and so $R(x+y)J_1J_2 = (0)$. It follows that $R(x+y) \in V_1$ and so $x+y \in P_1$. Thus P_1 is an ideal of R. Similarly, P_2 is an ideal of R. Let $xy \in P_1$ and suppose that $x \notin P_1$. So there exist vertices $I \in V_1$, $J \in V_2$ such that $xy \in I$ and $IJ = (0)$. We claim that $yJ = (0)$. For see this let $yJ \neq (0)$. Since $\mathbb{AG}(R)$ is bipartite and R is reduced, $IyJ = (0)$ implies that $yJ \in V_2$ and also $RxyJ = (0)$, implies that $Rx \in V_1$. It follows that $x \in P_1$, a contradiction. Therefore $yJ = (0)$ and so $Ry \in V_1$ i.e., $y \in P_1$. Thus P_1 is a prime ideal of R. Similarly, P_2 is a prime ideal of R. Now let $0 \neq t \in P_1 \cap P_2$. Then there exist $I \in V_1$, $J \in V_2$ such that $t \in I \cap J$. Also, there exist $L \in V_1$, $K \in V_2$ such that $IK = (0)$ and $JL = (0)$. It follows that $RtK = (0)$ and $RtL = (0)$. Since $\mathbb{AG}(R)$ is bipartite and R is reduced, $Rt \in V_1 \cap V_2 = \emptyset$, a contradiction. Therefore $P_1 \cap P_2 = (0)$.
Case 2: The ring R is not reduced. Assume $x^2 = (0)$, where $0 \neq x \in R$. Without loss of generality we can assume that $Rx \in V_1$. We claim that either Rx is a minimal ideal of R or for each $0 \neq z \in R$ such that $Rz \subsetneq Rx$, Rz is a minimal ideal of R. For see this let $(0) \neq Rz_1 \subsetneq Rx$, $(0) \neq Rz_1 \subset Rx$,$(0) \neq Rz_2 \subset Rz_1$ for some $z_1, z_2 \in R$. Since $(Rx)^2 = (0)$, $Rz_1Rx = Rz_2Rx = (0)$, and so Rz_1, $Rz_2 \in V_2$ since $\mathbb{AG}(R)$ is bipartite. But $Rz_1Rz_2 \subseteq (Rx)^2 = (0)$, a contradiction. Thus without loss of generality we can assume that Rx is a minimal ideal of R. Thus $P = Ann(x)$ is a maximal ideal of R and also $Rx \subseteq P$. We claim that $V_1 = \{Rx\}$.
Subcase 1: $P = Rx$. Then P is both minimal and maximal. Let $I \in V_1$ such that $I \neq Rx$. Then there exists $L \in V_2$ such that $IL = (0)$. Since P is prime and $I \nsubseteq P$, $L \subseteq P$ i.e., $L = P = Rx$, a contradiction.

Subcase 2: $P \neq Rx$. Since $PRx = (0)$ and $\mathbb{AG}(R)$ is bipartite, $P \in V_2$. Now let $V_1 \neq \{Rx\}$ and $L \in V_1 \setminus \{Rx\}$. If $LP = (0)$, $LRx \subseteq LP = (0)$, a contradiction. Thus $LP \neq (0)$ and since $\mathbb{AG}(R)$ is connected, there exists $K \in V_2$ such that $K \neq P$ and $LK = (0)$. It follows that $L \cap P \neq (0)$, $(L \cap P)Rx = (0)$. Therefore $L \cap P \in V_2$. But $(L \cap P)K = (0)$, a contradiction. Thus in each case $V_1 = \{Rx\}$ and it follows that $\mathbb{AG}(R)$ is a star graph.
(4) $\Rightarrow$ (3). Clearly if $\mathbb{AG}(R)$ is a star graph, then $\mathbb{AG}(R)$ is a complete bipartite graph. Thus we assume that there exist two distinct prime ideals P_1 and P_2 of R such that $P_1 \cap P_2 = (0)$. We claim that $Z(R) = P_1 \cup P_2$. It is clear that $P_1 \cup P_2 \subseteq Z(R)$. On the other hand, if $x \in Z(R) \setminus P_1 \cup P_2$, then there exists $0 \neq y \in R$ such that $xy = 0$, and hence $y \in P_1 \cap P_2 = (0)$, which is a contradiction. Thus $Z(R) = P_1 \cup P_2$ and so by Prime Avoidance Theorem (see [8, Theorem 3.61], for every $I \in \mathbb{A}^*(R)$, either $I \in P_1$ or $I \in P_2$. Also if $I_1, I_2 \subseteq P_1$, $I_1 I_2 \neq (0)$. Because otherwise $I_1 I_2 \subseteq P_2$ and so $I_1 \subseteq P_2$ or $I_2 \subseteq P_2$. Similarly for each two ideal included in P_2. Now, let f be a coloring assigns to each ideal I of $\mathbb{A}(R)$ color $i, i = 1, 2$ if $I \subseteq P_i, i = 1, 2$. So by above argument, $\mathbb{AG}(R)$ is bipartite. On the other hand, for vertices I, J such that $I \subseteq P_1$ and $J \subseteq P_2$, we have $IJ \subseteq P_1 \cap P_2 = (0)$. Therefore, $\mathbb{AG}(R)$ is a complete bipartite graph. $\square$

Corollary 2.4. *Let R be an Artinian ring. Then the following statements are equivalent:*

(1) $\chi(\mathbb{AG}(R)) = 2$.
(2) $\mathbb{AG}(R)$ *is a bipartite graph.*
(3) $\mathbb{AG}(R)$ *is a complete bipartite graph.*
(4) *Either $R \cong F_1 \times F_2$ for some fields F_1 and F_2 or R is a local ring such that $\mathbb{AG}(R)$ is a star graph.*

Proof. By Theorem 2.3, (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).
(2) $\Rightarrow$ (4). Assume that $\mathbb{AG}(R)$ is a bipartite graph with parts V_1 and V_2. Note that since R is Artinian, $R = R_1 \oplus ... \oplus R_n$ for some integer $n \in \mathbb{N}$ and some local Artinian rings $R_1, ..., R_n$. Suppose that $n > 2$. Since $(R_1 \oplus 0 \oplus ... \oplus 0)(0 \oplus R_2 \oplus 0 \oplus ... \oplus 0) = (0)$, and $\mathbb{AG}(R)$ is a bipartite graph, $R_1 \oplus 0 \oplus ... \oplus 0$ and $0 \oplus R_2 \oplus 0 \oplus ... \oplus 0 = (0)$ are in different parts of vertices, say in V_1 and V_2, respectively. Now $0 \oplus ... \oplus R_n$ is adjacent to both above vertices and so $0 \oplus ... \oplus R_n \in V_1 \cap V_2 = \emptyset$, a contradiction. Thus $n \leq 2$. Also Theorem 2.3 implies that either there exist two distinct prime ideals P_1 and

P_2 of R such that $P_1 \cap P_2 = (0)$ or $\mathbb{AG}(R)$ is a star graph. Let $n = 2$ and suppose that there exist two distinct prime ideals P_1 and P_2 of R such that $P_1 \cap P_2 = (0)$. Note that $P_1 = Q_1 \oplus Q_2$ for some ideals Q_1 of R_1 and Q_2 of R_2. But $(0 \oplus R_2)(Q_1 \oplus Q_2) = (0)$ which implies $R_2 Q_2 = (0)$ and from there $Q_2 = (0)$. Now, since $(R_1 \oplus (0))((0) \oplus R_2) = (0) \subseteq P_1$, we should have $(R_1 \oplus (0)) \subseteq P_1$ and so $Q_1 = R_1$. In other words, R_1 is a field. Similarly, we can prove that R_2 is a field, too. If $n = 2$ and $\mathbb{AG}(R)$ is a star graph, say with $|V_1| = 1$. It is evident that the only vertex of V_1 is $R_1 \oplus (0)$. Also, since R_2 is an Artinian local ring, $Z(R_2) = J(R_2)$ is an ideal of R_2. But $(0) \oplus R_2 \in V_2$ and so $R_2 \subseteq Z(R_2)$, i.e., R_2 is a field. Thus anyway, if $n = 2$, $R = F_1 \oplus F_2$. Let $n = 1$. Then $Z(R) = J(R)$. If there exist two distinct prime ideals P_1 and P_2 of R such that $P_1 \cap P_2 = (0)$, then $R \cong R/P_1 \oplus R/P_2$ and since R is Artinian, R/P_1 and R/P_2 are fields and we are done.
(4) $\Rightarrow$ (2) is clear. $\square$

In [4, Corollary 2.4], it is shown that for a reduced ring R, $\mathbb{AG}(R)$ is a star graph if and only if $R \cong F \oplus D$, where F is a field and D is an integral domain. It follows that R has exactly two minimal primes $F \times (0)$ and $(0) \times D$. Also, for a reduced ring R, $\chi(\Gamma(R)) = 2$ if and only if R has exactly two minimal primes (see [3, Theorem 3.8]). Thus by using these facts and Theorem 2.3, we have the following interesting result for reduced rings.

Corollary 2.5. *Let R be a reduced ring. Then the following statements are equivalent:*

(1) $\chi(\mathbb{AG}(R)) = 2$.
(2) $\chi(\Gamma(R)) = 2$.
(3) $\mathbb{AG}(R)$ *is a bipartite graph.*
(4) $\mathbb{AG}(R)$ *is a complete bipartite graph.*
(5) R *has exactly two minimal primes.*

One can easily see that for every ring R,

$$\chi(\Gamma(R)) = 2 \implies \chi(\mathbb{AG}(R)) = 2.$$

But the following example shows that in general, if R is not reduced,

$$\chi(\mathbb{AG}(R)) = 2 \not\Longrightarrow \chi(\Gamma(R)) = 2.$$

Example 2.6. Let $R = \mathbb{Z}_{p^3}$ where p is a positive prime number. Then it is easy to check that $\chi(\mathbb{AG}(R)) = 2$, but $\chi(\Gamma(R)) \geq p$.

The following proposition shows that if $\chi(\Gamma(R))$ is finite, then $\chi(\mathbb{AG}(R))$ is also finite.

Proposition 2.7. *Let R be a ring. If $\mathbb{AG}(R)$ has an infinite clique, then $\Gamma(R)$ has an infinite clique.*

Proof. Suppose that C is an infinite clique of $\mathbb{AG}(R)$. Let Υ be the set of all vertices I of C such that $I^2 = (0)$.
Case 1: If $\bigcup \Upsilon$ is infinite, then for every $x, y \in \Upsilon$, $xy = (0)$ and so $\Gamma(R)$ has an infinite clique.
Case 2: If $\bigcup \Upsilon$ is finite. Then there is finitely many vertices I of C such that $I^2 = (0)$. Thus $C \setminus \Upsilon$ is infinite. We claim that in this case, for every $I \in C \setminus \Upsilon$, $I \nsubseteq \bigcup_{\phi\in\Phi} J_\phi$, where $\{J_\phi | \phi \in \Phi\}$ is an arbitrary subset of $C \setminus \Upsilon$ such that $I \notin \{J_\phi | \phi \in \Phi\}$. Because if $I \subseteq \bigcup_{\phi\in\Phi} J_\phi$, then $I^2 = (0)$, a contradiction. Now, let $\{I_i | i \in \mathbb{N}\}$ be a subset of distinct vertices of $C \setminus \Upsilon$. Let $x_1 \in I_1$. Since $I_2 \nsubseteq I_1$, there exists $x_2 \in I_2 \setminus I_1$ such that $x_2 \neq x_1$. Again, since $I_3 \nsubseteq (I_1 \cup I_2)$, there exists $x_3 \in I_3 \setminus (I_1 \cup I_2)$ such that $x_3 \neq x_1, x_2$. Continuing this way, we can get an infinite clique of $\Gamma(R)$. $\square$

We are going to characterize reduced rings R for which $\chi(\mathbb{AG}(R))$ is finite. To see that, we need some prefaces.

Lemma 2.8. *Let R be a reduced ring such that $\mathbb{AG}(R)$ does not have an infinite clique. Then R is has ACC on ideals of the form $Ann(I)$ where I is an ideal of R.*

Proof. Let $Ann(I_1) \subset Ann(I_2) \subset Ann(I_3) \subset ...$ be a chain in $\mathbb{A}(R)$. Clearly, $I_i Ann(I_{i+1}) \neq (0)$ for each $i \geq 1$. Thus for each $i \geq 1$, there exists $x_i \in I_i$ such that $x_i Ann(I_{i+1}) \neq (0)$. Let $J_i = x_i Ann(I_{i+1}), i = 2, 3, ...$ Then if $i \neq j$, $J_i \neq J_j$ because if $J_i = J_j$, then $J_i^2 = J_j^2 = 0$, contradiction. $\square$

Lemma 2.9. ([3, Lemma 3.6.]) *Let $P_1 = Ann(x_1)$ and $P_2 = Ann(x_2)$ be two distinct elements of $Spec(R)$. Then we have $x_1x_2 = 0$.*

Theorem 2.10. *For a reduced ring R, the following statements are equivalent:*
1) $\chi(\mathbb{AG}(R))$ *is finite.*
2) $cl(\mathbb{AG}(R))$ *is finite.*
3) $\mathbb{AG}(R)$ *does not have an infinite clique.*
4) *R has finite number of minimal prime ideals.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is clear.
(4) $\Rightarrow$ (1). Let $(0) = P_1 \cap P_2 \cap ... \cap P_k$, where $P_1, ..., P_k$ are prime ideals. Define a coloring f on $\mathbb{A}(R)^*$ by putting $f(J) = min\{n \in \mathbb{N} : J \nsubseteq P_n\}$ for $J \in \mathbb{A}(R)^*$. Then $\chi(\mathbb{AG}(R)) \leq k$.
(3) $\Rightarrow$ (4). Suppose that R doesn't have an infinite clique. So by Lemma 2.8, R has ACC on ideals of the form $Ann(I)$, $I \in \mathbb{I}(R)$. Thus the set $\{Ann(x) : 0 \neq x \in R\}$ has maximal ideals, and it is easy to see that these are prime ideals of R. Let $Ann(x_\lambda)$, $\lambda \in \Lambda$ be the different maximal members of the family $\{Ann(x) : 0 \neq x \in R\}$. By Lemma 2.9, the index set Λ is finite. Pick $x \in R$, $x \neq 0$. Then $Ann(x) \subseteq Ann(x_\lambda)$ for some $\lambda \in \Lambda$. Now $x_\lambda x \neq 0$ because otherwise, $x_\lambda^2 = 0$, which is a contradiction. Thus $x \notin Ann(x_\lambda)$. This means $\bigcap_{\lambda \in \Lambda} Ann(x_\lambda) = (0)$. $\square$

Now we are in position to prove Conjecture 0.1 in the case of reduced rings. In fact the next corollary is an immediate conclusion of Theorem 2.10.

Corollary 2.11. *Let R be a reduced ring. Then $\chi(\mathbb{AG}(R)) = cl(\mathbb{AG}(R))$. Moreover, if $\chi(\mathbb{AG}(R))$ is finite, then R has a finite number of minimal primes, and if n is this number, then*

$$\chi(\mathbb{AG}(R)) = cl(\mathbb{AG}(R)) = n.$$

Adding Corollary 2.11, and [3, Theorem 3.9.] to Theorem 2.10, the next corollary is indisputable.

Corollary 2.12. *Let R be a nonzero reduced ring. Then $\chi(\mathbb{AG}(R))$ is finite if and only if $\chi(\Gamma(R))$ is finite. Furthermore, if they are finite, then*

$$\chi(\mathbb{AG}(R)) = \chi(\Gamma(R)).$$

Now, we will have a characterization of Noetherian rings R for which $\chi(\mathbb{AG}(R)) < \infty$. Before that, we need the following lemma.

Lemma 2.13. *Let R be a ring. If $cl(\mathbb{AG}(R))$ is finite, then*

(i) *for every nonzero ideal I of R with $I^2 = (0)$, I has finite number of R-submodules; and*
(ii) *for every element $a \in Nil(R)$, Ra has finite number of R-submodules.*

Proof. (i) Evident.
(ii) Suppose that $a \in Nil(R)$, $a^n = (0)$ and n is the least integer with this property. If $n = 2$, (i) shows that Ra has finite number of R-submodules. Suppose that $n > 2$. Let $k = n/2$ if n is an even integer and $k = (n+1)/2$, if n is an odd one, and let $b = a^k$. Then $(Rb)^2 = (0)$. Thus by (i), Rb has finite number of R-submodules. Assume that Ra has infinite number of R-submodules. Define

$$\varphi : Rb \to Ra$$
$$rb \mapsto ra$$

It is easy to check that φ is a ring epimorphism and so $Rb/ker(\varphi) \cong Ra$. Since Ra has infinite number of R-submodules, so $Rb/ker(\varphi)$ and from there Rb has infinite number of R-submodules, a contradiction. $\square$

Theorem 2.14. *Let R be a Noetherian ring. Then $cl(\mathbb{AG}(R))$ is finite if and only if for every ideal I of R with $I^2 = (0)$, I has finite number of R-submodules.*

Proof. $(\Rightarrow)$ is by Lemma 2.13 (i).
$(\Leftarrow)$ Suppose that for every ideal I of R with $I^2 = (0)$, I has finite number of R-submodules. Let C be the largest clique in $\mathbb{AG}(R)$, and let Υ be the set of all vertices I of C with $I^2 = (0)$. Then $J = \sum_{I \in \Upsilon} I$ is again a vertex of C and $J^2 = (0)$. So by our hypothesis, J has finite number of R-submodules. But if $I \in \Upsilon$, every R-submodule of I is an R-submodule of J. Thus for every $I \in \Upsilon$, I have finite number of R-submodules and specially Υ has finite element. We claim that $C \setminus \Upsilon$ has infinite member, too. Suppose that $\{I_1, I_2, ...\}$ is an infinite subset of $C \setminus \Upsilon$. Consider the chain $I_1 \subseteq I_1 + I_2 \subseteq I_1 + I_2 + I_3 \subseteq ....$ Since R is Noetherian, there exists $n \in \mathbb{N}$, such that $I_1 + ... + I_n = I_1 + ... + I_n + I_{n+1}$, i.e. $I_{n+1} \subseteq I_1 + ... + I_n$. Since by our choice of C, $I_1 + ... + I_n \in C$, I_{n+1} is adjacent to $I_1 + ... + I_n$ and thus $I_{n+1}^2 = (0)$, a contradiction. Thus C has finite number of vertices and from there, $cl(\mathbb{AG}(R))$ is finite. $\square$

By Theorem 2.10, one may naturally guess the following statements for the general case.We close this section with this conjecture.

Conjecture 2.15. *Let R be a ring. If $\mathbb{AG}(R)$ does not have an infinite clique, then $cl(\mathbb{AG}(R))$ is finite.*

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